J. DIFFERENTIAL GEOMETRY
11 (1976) 401-409

THE HIGHER HOMOTOPY GROUPS OF LINKS

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1. Introduction

In this paper we generalize the result of Andrews and Lomonaco [2] and McCallum [7] in which the second homotopy group of a 1-spun classical knot and link respectively were calculated to obtain results about k-spinning higher dimensional links. We take the approach of Lomonaco [6] using Reidemeister homotopy chains [9].

In particular we prove the following theorem.

Theorem 1.1. *If L_μ^{n+k} is an $(n+k)$-dimensional link of multiplicity μ obtained by k-spinning an n-dimensional ball configuration $K_\mu^n \subset B^{n+2}$ about the sphere $S^{n+1} = \partial B^{n+2}$ with $B^{n+2} - K_\mu^n$ aspherical, and*

$$(x_1, x_2, \cdots, x_m : r_1, r_2, \cdots, r_p)$$

is a presentation of $\Pi_1(S^{n+k+2} - L_\mu^{n+k})$ with $x_1, x_2, \cdots, x_{\mu_1}$ $(0 \leq \mu_1 \leq m)$ the images of the generators of $\Pi_1(S^{n+1} - K_\mu^n)$ under the inclusion map, then

$$\Pi_i(S^{n+k+2} - L_\mu^{n+k}) = 0 \qquad (1 < i \leq k) ,$$

and

$$\left(x^*_{\mu_1+1}, x^*_{\mu_1+2}, \cdots, x^*_m : \sum_{j=\mu_1+1}^{m} (\partial r_i / \partial x_j) x^*_j\right)$$

is a presentation of $\Pi_{k+1}(S^{n+k+2} - L_\mu^{n+k})$ as a left $Z\Pi_1$-module. We then apply this algorithm to particular well known links and, in fact, obtain yet another proof of the main result found in [1].

2. Preliminary results

Definition 2.1. A ball configuration

$$K_\mu^n : B_1^n \cup B_2^n \cup \cdots \cup B_\mu^n \subset B_\mu^{n+2}$$

is a piecewise-linear proper embedding of the disjoint union of μ copies of B^n in B^{n+2}.

Received November 11, 1974, and, in revised form, January 27, 1975.

Lemma 2.2. *In $Y = B^{n+2} - K^n_\mu$ there exists an $(n + 1)$-dimensional C. W. complex K consisting of*

$$\begin{array}{rl} 0\text{-}cell: & x^0_1 \, , \\ 1\text{-}cells: & x^1_1, x^1_2, \cdots, x^1_{m_1} \, , \\ \cdots & \cdots \\ (n+1)\text{-}cells: & x^{n+1}_1, x^{n+1}_2, \cdots, x^{n+1}_{m_{n+1}} \end{array}$$

such that $(y, \partial Y)$ deformation retracts onto $(k, \partial K)$ where $\partial K = K \cap \partial Y$, which consists of

$$\begin{array}{rl} 0\text{-}cell: & x^0_1 \, , \\ 1\text{-}cells: & x^1_1, x^1_2, \cdots, x^1_{\mu_n} \, , \\ \cdots & \cdots \\ (n+1)\text{-}cells: & x^n_1, x^n_2, \cdots, x^n_{\mu_n} \, , \end{array}$$

Proof. Let T be a triangulation of R^{n+2} which has K^n_μ as a subcomplex and has ∂B^{n+2} as a subcomplex of its dual triangulation $T^\#$ where B^{n+2} is an $(n + 2)$-simplex. Adjoin to K^n_μ any edge of T, which lies within B^{n+2} and meets K^n_μ at one end point only. Continue this process as long as possible which is only a finite number of times, always adjoining an edge of T within B^{n+2} which meets the previously constructed complex at one end point only. There results an n-dimensional subcomplex K'', which meets ∂B^{n+2} at $K^n_\mu \cap \partial B^{n+2}$ and has K^n_μ as a deformation retract. Let K' be the subcomplex of $T^\#$ made up of cells which lie in B^{n+2} and do not meet K''. Then K' is a complex of dimension $\leq n + 1$, and K' is a deformation retract of $B^{n+2} - K''$ and hence of $B^{n+2} - K^n_\mu$. Furthermore, $K' \cap \partial B^{n+2}$ is ∂B^{n+2} minus several of its open faces. Therefore a cell complex K of the required type is obtained by shrinking to a point the remaining cells plus a suitable maximal tree of K''.

It now follows that $\Pi_1(Y) = (x^1_1, x^1_2, \cdots, x^1_{m_1} : x^2_1, x^2_2, \cdots, x^2_{m_2})$ where we are identifying the elements of $\Pi_1(Y)$ with their carriers in K.

Let $\tilde{K}$ be the universal cover of K. In $\tilde{K}$ we have

$$\begin{array}{rll} \text{0-cells}: & g\tilde{x}^0_1 \, , & g \in \Pi_1(Y) \, , \\ \text{1-cells}: & g\tilde{x}^1_i & (1 \leq i \leq m_1) \, , \\ \cdots & \cdots & \\ (n+1)\text{-cells}: & g\tilde{x}^{n+1}_i & (1 \leq i \leq m_{n+1}) \, . \end{array}$$

We consider the Reidemeister homotopy chain complex

$$0 \to C_{n+1}(\tilde{K}) \to \cdots \to C_0(\tilde{K})$$

with boundary homorphisms given by

$$\partial_i(g\tilde{x}^i_j) = g\partial_i(\tilde{x}^i_j) ,$$

$(0 \leq i \leq n+1,\ 1 \leq j \leq m_i)$, (see [9]).

3. k-spinning

Definition 3.1. One obtains the $(n+k)$-dimensional link L^{n+k}_μ of multiplicity μ by k-spinning K^n_μ as follows:

$$S^{n+k+2} = (S^k \times B^{n+2}) \cup (D^{k+1} \times B^{n+2})$$

identified along

$$S^k \times \partial B^{n+2} = \partial D^{k+1} \times \partial B^{n+2} ,$$

and

$$S^{n+k}_i = (S^k \times B^n_i) \cup (D^{k+1} \times \partial B^n_i)$$

identified along

$$S^k \times \partial B^n_i = \partial D^{k+1} \times \partial B^n_i$$

(see [4] and [11]).

If $n = k = 1$, then this definition is equivalent to the classical spinning technique of Artin [3]. We have the following lemma due to Artin [3] and Summers [11].

Lemma 3.2. *Suppose L^{n+k}_μ is obtained by k-spinning K^n_μ. Let $X = S^{n+k+2} - L^{n+k}_\mu$, and $Y = B^{n+2} - K^n_\mu$. Then*

$$\Pi_1(X) = \Pi_1(Y) .$$

Proof. See [11].

We now k-spin Y to obtain X as follows:

$$X = (S^k \times Y) \cup (D^{k+1} \times \partial Y)$$

identified along

$$S^k \times \partial Y = \partial D^{k+1} \times \partial Y .$$

Lemma 3.3. *X will deformation retract onto an $(n+k+1)$-dimensional C. W. complex K^* with the following cells:*

Type I. *Cells obtained from the deformation of Y:*

$$\begin{aligned} &0\text{-}cell: && x_1^0\ , \\ &1\text{-}cells: && x_1^1, x_2^1, \cdots, x_{m_1}^1\ , \\ &\cdots\cdots && \\ &(n+1)\text{-}cells: && x_1^{n+1}, x_2^{n+1}, \cdots, x_{m_{n+1}}^{n+1}\ . \end{aligned}$$

Type II. *Cells obtained by k-spinning cells of type* I:

$$\begin{aligned} &k\text{-}cells: && x_1^{0*}\ , \\ &(k+1)\text{-}cells: && x_1^{1*}, x_2^{1*}, \cdots, x_{m_1}^{1*}\ , \\ &\cdots\cdots && \\ &(n+k+1)\text{-}cells: && x_1^{n+1*}, x_2^{n+1*}, \cdots, x_{m_{n+1}}^{n+1*}\ . \end{aligned}$$

Type III. *Cells obtained by the deformation of the "plug",* $D^{k+1} \times \partial Y$:

$$\begin{aligned} &(k+1)\text{-}cells: && x_1^{0**}\ , \\ &(k+2)\text{-}cells: && x_1^{1**}, x_2^{1**}, \cdots, x_{\mu_1}^{1**}\ , \\ &\cdots\cdots && \\ &(n+k+1)\text{-}cells: && x_1^{n**}, x_2^{n**}, \cdots, x_{\mu_n}^{n**}\ . \end{aligned}$$

The proof of Lemma 3.3 follows from the definition of k-spinning and Lemma 2.2.

Let $\tilde{K}^*$ be the universal cover of K^*. Then the cell structure of $\tilde{K}^*$ is given by the following three types:

$$\begin{aligned} &\text{Type I}'. && 0\text{-cells}: && g\tilde{x}_1^0\ , && g \in \Pi_1(X) \simeq \Pi_1(Y)\ , \\ & && 1\text{-cells}: && g\tilde{x}_i^1 && (1 \le i \le m_1)\ , \\ & && \cdots\cdots && && \\ & && (n+1)\text{-cells}: && g\tilde{x}_i^{n+1} && (1 \le i \le m_{n+1})\ . \\ &\text{Type II}': && k\text{-cells}: && g\tilde{x}_i^{0*}\ , && g \in \Pi_1(X)\ , \\ & && (k+1)\text{-cells}: && g\tilde{x}_i^{1*} && (1 \le i \le m_1)\ , \\ & && \cdots\cdots && && \\ & && (n+k+1)\text{-cells}: && g\tilde{x}_i^{n+1*} && (1 \le i \le m_{n+1})\ . \\ &\text{Type III}'. && (k+1)\text{-cells}: && g\tilde{x}_1^{0**}\ , && g \in \Pi_1(X)\ , \\ & && (k+2)\text{-cells}: && g\tilde{x}_i^{n**} && (1 \le i \le \mu_1)\ , \\ & && \cdots\cdots && && \\ & && (n+k+1)\text{-cells}: && g\tilde{x}_i^{n**} && (1 \le i \le \mu_n)\ . \end{aligned}$$

We now observe that the boundary homorphisms of the Reidemeister homotopy chain complex of $\tilde{K}^*$,

$$0 \to C_{n+k+1}(\tilde{K}^*) \to \cdots \to C_0(\tilde{K}^*) ,$$

are given by

Type I′: $\partial_i^*(g\tilde{x}^i) = g\partial_i^*(\tilde{x}^i) = g\partial_i(\tilde{x}^i)$,

Type II′: $\partial_i^*(g\tilde{x}^{i*}) = g\partial_i^*(\tilde{x}^{i*}) = g(\partial_i\tilde{x}^i)^*$,

where if

$$\partial_i\tilde{x}^i = \sum_{j=1}^{m_{i-1}} g_j\tilde{x}_j^{i-1} ,$$

then

$$(\partial_i\tilde{x}^i)^* + \sum_{j=1}^{m_{i-1}} g_j\tilde{x}_j^{i-1*} ,$$

(see Fig. 3.1),

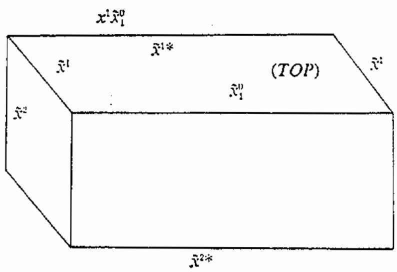

Fig. 3.1

Type III′: $\partial_i^*(g\tilde{x}^{i**}) = g\partial_i^*(\tilde{x}^{i**}) = g\tilde{x}^{i*}$,

as

$$\partial^*(\tilde{x}^{i**}) = \partial(D^{k+1} \times \tilde{x}^i) = (S^k \times \tilde{x}^i) = \tilde{x}^{i*} ,$$

$H_i(\tilde{K}^*) = 0$ $(1 < i \leq k)$, and

$$H_{k+1}(\tilde{K}^*) = (x_{\mu_1+1}^{1*}, x_{\mu_1+2}^{1*}, \cdots, x_{m_1}^{1*} : \partial_2^* x_1^{2*}, \cdots, \partial_2^* x_{m_2}^{2*}) .$$

In particular, ∂_2^* is given by the Fox free derivatives [5]. Hence by the Hurewiz theorem

$$\Pi_n(X) = \Pi_n(K^*) = \Pi_n(\tilde{K}^*) = H_n(\tilde{K}^*) \qquad (1 < n \leq k+1)$$

as a $Z\Pi_1$-module, and our theorem is proved for one particular presentation of $\Pi_1(X)$. The general theorem will follow from the following two lemmas which show that the Tietze I and II operations on the presentation of $\Pi_1(X)$ induce Tietze I and II operations on $\Pi_{k+1}(X)$ as a $Z\Pi_1$-module.

Lemma 4.3. *If a relation s is a consequence of*

$$F = (r_1, r_2, \cdots, r_m) ,$$

then $\partial s/\partial x$ *is a consequence of*

$$\partial F/\partial x = (\partial r_1/\partial x, \partial r_2/\partial x, \cdots, \partial r_m/\partial x) ,$$

where s is the relation.

Proof. In ZF we have

$$\begin{aligned}\partial s/\partial x &= \partial\Big(\prod_{k=1}^{p} u_k r_{i_k}^{a_k} u_k^{-1}\Big)/\partial x \\ &= \partial(u_1 r_{i_1}^{a_1} u_1^{-1})/\partial x + (u_1 r_{i_1}^{a_1} u_1^{-1})\partial(u^2 r_{i_2}^{a_2} u_2^{-1})/\partial x \\ &\quad + \cdots + \prod_{k=1}^{p} (u_k r_{i_k}^{a_k} u_k^{-1})\partial(u_p r_{i_p}^{a_p} u_p^{-1})/\partial x ,\end{aligned}$$

but as $r_i \to 1$ in $Z\Pi_1$ and identifying $\partial s/\partial x$ with its image in $Z\Pi_1$ we obtain

$$\partial s/\partial x = \sum_{k=1}^{p} \partial(u_k r_{i_k}^{a_k} u_k^{-1})/\partial x .$$

Since

$$\frac{\partial}{\partial x}(u_k r_{i_k}^{a_k} u_k^{-1}) = \frac{\partial u_k}{\partial x} + \frac{u_k(r_{i_k}^{a_k} - 1)}{r_{i_k} - 1}\frac{\partial r_{i_k}}{\partial x} - u_k r_{i_k}^{a_k} u_k^{-1}\frac{\partial u_k}{\partial x}$$

and

$$(r_{i_k}^{a_k} - 1)/(r_{i_k} - 1) = a_k ,$$

we have

$$\partial(u_k r_{i_k}^{a_k} u_k^{-1})/\partial x = a_k u_k \partial r_{i_k}/\partial x ,$$

so that

$$\partial s/\partial x = \sum_{k=1}^{p} a_k u_k \partial r_{i_k}/\partial x .$$

Lemma 4.4. *The Tietze* II *operation on* $\Pi_1(Y)$ *induces a Tietze* II *or the iedntity operation on* $\Pi_{k+1}(X)$ *as a left* $Z\Pi_1$*-module depending on whether e is in the interior of Y or on its boundary.*

Proof. Consider the Tietze II operation

$$\text{II}: \quad (\bar{x}, \bar{r}) \to (\bar{x} \cup y : \bar{r} \cup ye^{-1}) ,$$

where y is a member of the underlying set of generators not contained in $\bar{x}$. Suppose that e is not on the boundary of Y, then it remains to show that

$$\Pi'_{k+1} = \Bigl(x^*_{\mu_1+1}, x^*_{\mu_1+2}, \cdots, x^*_m, y^* : \sum_{j=\mu_1+1}^{m} \frac{\partial r_i}{\partial x_j}(x^*_j + y^*), \sum_{j=\mu_1+1}^{m} \frac{\partial ye^{-1}}{\partial x_j} x^*_j + \frac{\partial ye^{-1}}{\partial y} y^*\Bigr)$$

is obtained from

$$\Pi_{k+1} = \Bigl(x^*_{\mu_1+1}, x^*_{\mu_1+2}, \cdots, x^*_m : \sum_{j=\mu_1+1}^{m} (\partial r_i/\partial x_j) x^*_j\Bigr)$$

by a Tietze II operation. But as r and e do not contain any factor equal to y as a member of the free group on elements of $\bar{x}$, we have that $\partial r_i/\partial x_j = 0$ $(i = 1, 2, \cdots, p)$ and further that

$$\sum_{j=\mu_1+1}^{m} \frac{\partial ye^{-1}}{\partial x_j} x^*_j + \frac{\partial ye^{-1}}{\partial y} y^* = y \sum_{y=\mu_1+1}^{m} \frac{\partial e^{-1}}{\partial x_j} x^*_j + y^* ,$$

and the result follows. If, on the other hand, e were on the boundary, then $(\partial e^{-1}/\partial x_j) = 0$ for all j, and hence Π'_{k+1} has the same presentation as Π_{k+1}.

5. Application

In particular we note that if we k-spin a 1-dimensional ball configuration, which is geometrically unsplittable and intersects ∂B^3, then the complex K is always aspherical (see [8]), and further the 2-dimensional C. W. complex K will have one vertex p, n edges $x_1, x_2, \cdots, x_n$ and $n - \mu$ faces $r_{\mu+1}, r_{\mu+2}, \cdots, r_n$ as ∂Y is a surface of genus μ, so that

$$\chi(K) = \chi(S^3 - K^1_\mu) = \tfrac{1}{2}\chi(\partial(S^3 - K^1_\mu)) = 1 - \mu .$$

Application 5.1. Two linked knotted two-spheres in the four-sphere.

We obtain yet a third proof (the first given in Van Kampen [13] and the second given in Shinohara and Sumners [10]) that the two unknotted 2-spheres obtained by 1-spinning the ball configuration in Fig. 5.1 are not isotopically splittable as

$$\Pi_1(B^3 - K^1_2) = (a, b, x : xax^{-1}b^{-1}axa^{-1}b^{-1}ax^{-1}a^{-1}b) .$$

Also

$$\Pi_2(S^4 - L^2_2) = (X : (1 - b^{-1}a - xax^{-1} + xax^{-1}b^{-1}a)X) ,$$

and $\Pi_2(S^4 - L_2^2)$ is nontrivial as it can be mapped onto the integers. However, if S_1^2 and S_2^2 were isotopically splittable, then $S^4 - L_2^2$ would deformation retract to $S^1 \vee S^1 \vee S^3$, and hence $\Pi_2(S^4 - L_2^2) = 0$.

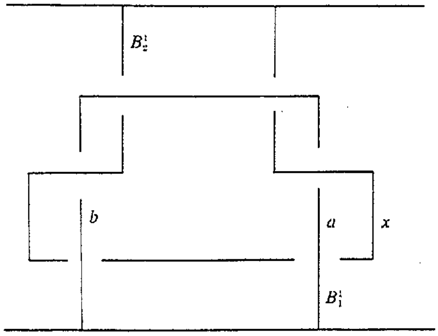

Fig. 5.1

Application 5.2. An Unknotted two-sphere linked with a knotted two-sphere in the four-sphere.

We give a proof that k-spinning the ball configuration as given in Fig. 5.2 is not isotopically splittable. Artin [3] originally showed this to be true for

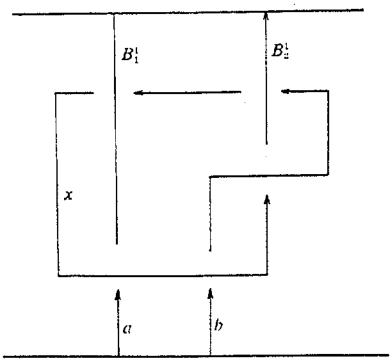

Fig. 5.2

1-spinning, and later Andrews and Curtis [1] showed that the 2-spheres obtained by 1-spinning K_2^1 were not homotopically splittable. We note that if the two $(k+1)$-spheres obtained by k-spinning were isotopically splittable, then

$$\Pi_{k+1}(S^{k+3} - L_2^{k+1}) = 0 .$$

However,

$$\begin{aligned}\Pi_{k+1} \twoheadrightarrow \Pi_{k+1} \otimes_{Z\Pi_1} ZJ(t) &= (X: (t + t^{-2} - t^{-1})X) \\ &= (X: (t^3 - t - 1)X)\end{aligned}$$

$$= ZJ/(t^3 - t + 1)$$
$$= Z \otimes Z \otimes Z \quad \text{(see [12])} ,$$

where

$$\Pi_1(B^3 - K_2^1) = (a, b, x : x^{-1}b^{-1}xbaxa^{-1}b^{-1}) ,$$
$$\Pi_{k+1}(S^{k+3} - L_2^{k+1}) = (X : (bax^{-1} + x^{-1}b^{-1} - x^{-1})X) .$$

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